PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 9, Fasc. 2 (1988), p. 113–121

ON THE DOBRUSHIN'S HYPOTHESIS

BY

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Abstract. The central limit theorem for the stationary random processes under generalized mixing conditions is proved. The well-known Ibragimov's results are given as a special case of the theorem received.

1. In [3] Dobrushin has introduced certain weak dependence conditions for random fields, which form natural generalization of the known mixing conditions to be found in [4]. In the same paper Dobrushin has suggested that under these generalized mixing conditions it is possible to prove a central limit theorem which would contain the well-known results as special cases. Here we prove the Dobrushin hypothesis for 1-dimensional case.

2. Let X be a metric space with metric $\varrho(x, \tilde{x})$, $x, \tilde{x} \in X$, $\mathscr{B}$ be its Borel σ-algebra, and P and Q be the probability distribution on $\mathscr{B}$.

The quantity $R(P, Q) = \inf \mathrm{E}\,\varrho(\eta, \zeta)$, where the "inf" is taken over all 2-dimensional random vectors (η, ζ) which marginal distributions coincide with P and Q, respectively, is a metric on the space of probability distributions on $(X, \mathscr{B})$ and is called the *Wasserstein* or, sometimes, the *Kantorovich-Rubinstein distance* [6].

In [3] it is shown that if

$$\varrho(x, \tilde{x}) = \begin{cases} 1, & x \neq \tilde{x}, \\ 0, & x = \tilde{x}, \end{cases} \tag{1}$$

then

$$R(P, Q) = \sup_{B \in \mathscr{B}} |P(B) - Q(B)|,$$

i.e. R becomes the well-known variation metric.

If $X = R^k$, where k is any positive integer and

$$\varrho^{(k)}(x, \tilde{x}) = \sum_{i=1}^{k} |x_i - \tilde{x}_i|, \qquad x, \tilde{x} \in R^k,$$

then (cf. [5])

$$R(P, Q) = \int_{R^k} |F(x) - G(x)|\, dx,$$

where $F(x)$ and $G(x)$ are the distribution functions of P and Q, respectively.

Let $\{\xi_t\} = \{\xi_t, t \in \boldsymbol{Z}\}$ be a stationary process which takes values in the space X, where $\boldsymbol{Z}$ is the set of the integers, and let $\boldsymbol{P} = \{P_V, V \subset \boldsymbol{Z}\}$ be the set of its finite-dimensional distributions. Here every P_V is a probability measure on the σ-algebra of Borel subsets of the metric space $X^{|V|} = \{(x_1, \ldots, x_V), x_i \in X, i = 1, 2, \ldots, |V|\}$,

$$\varrho_V(x, \tilde{x}) = \sum_{t=1}^{|V|} \varrho(x_t, \tilde{x}_t), \qquad x, \tilde{x} \in X^{|V|}, \tag{2}$$

where $|V|$ denotes the number of points in a (finite) set V.

We say that a *random process* $\{\xi_t\}$ *satisfies the generalized strong mixing condition* (g.m.c.) if

$$R(P_{(-k,0)\cup(n,n+m)}, P_{(-k,0)} \times P_{(n,n+m)}) \leqslant \alpha_\varrho(n) \qquad \text{for any } k, m, n \in N, \tag{3}$$

where $\alpha_\varrho(n) \to 0$ as $n \to \infty$. Here (a, b) denotes the set of integers between a and b, $a < b$ $(a, b \in \boldsymbol{Z})$.

It is clear that if $X = R$ and the metric ϱ is discrete, i.e. coincides with (1), then (3) is the usual Rosenblatt strong mixing condition

$$|\mathrm{P}(AB) - \mathrm{P}(A)\,\mathrm{P}(B)| \leqslant \alpha(n) \tag{4}$$

for any $A \in \sigma(\xi_t, t \leqslant 0)$ and $B \in \sigma(\xi_t, t \geqslant n)$, $n = 1, 2, \ldots$ and $\alpha(n) \to 0$ as $n \to \infty$.

By changing the space X and the metric ϱ one can obtain various new mixing conditions. For instance, if $X = R$ and $\varrho(x, \tilde{x}) = |x - \tilde{x}|$, $x, \tilde{x} \in R$, then (3) reduces to

$$\int_{R^{k+m}} \Big|\mathrm{P}\Big(\bigcap_{t \in V_1 \cup V_2} (\xi_t < x_t)\Big) - \mathrm{P}\Big(\bigcap_{t \in V_1} (\xi_t < x_t)\Big)\mathrm{P}\Big(\bigcap_{t \in V_2} (\xi_t < x_t)\Big)\Big| \times \prod_{t \in V_1 \cup V_2} dx^t \leqslant \hat{\alpha}(n), \tag{5}$$

where $V_1 = (-k, 0)$, $V_2 = (n, n+m)$, $\hat{\alpha}(n) \to 0$ as $n \to \infty$ independently of $k, m \in N$.

We will use mixing conditions (4) and (5) to illustrate our general proposition.

Note that various conditions under which the random field satisfies g.m.c. have been presented in [3].

3. Let $f(x)$, $x \in X$, be a continuous function on (X, ϱ) and let $\tau^f(\gamma)$, $\gamma \in R_+$, denote the *continuity modulus* of f, i.e.

$$\tau^f(\gamma) = \sup_{(x,\tilde{x}):\varrho(x,\tilde{x})<\gamma} |f(x)-f(\tilde{x})|.$$

We say that the *process* $\{\xi_t\}$ *satisfies the central limit theorem* (CLT) *with function* f if, for any $s \in R$,

$$\lim_{n\to\infty} \mathrm{P}\Big(\big(\mathcal{D}\sum_{t=1}^{n} f(\xi_t)\big)^{-1/2} \sum_{t=1}^{n} \big(f(\xi_t)-\mathrm{E}f(\xi_t)\big) < s\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-u^2/2}\,du,$$

where $\mathcal{D}$ stands for the variance.

Our main result is as follows:

THEOREM. *Suppose that the stationary random process* $\{\xi_t\}$ *satisfies the g.m.c. and*:

1. *for some* $\delta > 0$, $E|f(\xi_t)|^{2+\delta} < \infty$ (*or, with probability* 1, $|f(\xi_t)| < C < \infty$);

2. *there exists a decreasing sequence* $\gamma_n \in R_+$, $\gamma_n \downarrow 0$ *as* $n \to \infty$, *such that* $\gamma_n^{-1}\alpha_\varrho(n) \downarrow \beta(n)$, $\beta(n) \downarrow 0$ *as* $n \to \infty$, *and*

$$\sum_{n=1}^{\infty} \tau^f(\gamma_n) < \infty, \qquad \sum_{n=1}^{\infty} \beta^{\delta/(2+\delta)}(n) < \infty \qquad \Big(or \sum_{n=1}^{\infty} \beta(n) < \infty\Big).$$

Then the series

$$\sigma_f^2 = \mathrm{E}\big(f(\xi_1)-\mathrm{E}f(\xi_1)\big)^2 + 2\sum_{n=2}^{\infty} \mathrm{E}\big(f(\xi_1)-\mathrm{E}f(\xi_1)\big)\big(f(\xi_n)-\mathrm{E}f(\xi_n)\big)$$

converges and, if $\sigma_f^2 \neq 0$, *then the process satisfies the CLT with the function* f.

Note that, in case of $X = R$ and discrete metric ϱ, the continuity modulus $\tau^f(\gamma)$, $\gamma \in R_+$, of any function f on X is equal to zero for $\gamma < 1$ and so the well-known Ibragimov's result [2] on CLT for stationary random processes becomes a special case of our theorem for $f(x) = x$, $x \in R$. It has been shown in [5] and [6] that these results of Ibragimov practically cannot be improved.

If $X = R$ and $\varrho(x, \tilde{x}) = |x-\tilde{x}|$, $x, \tilde{x} \in R$, $f(x) = x$, then $\tau^f(\gamma) = \gamma$, $\gamma \in R_+$, and we have the following

COROLLARY. *Suppose that the stationary random process* $\{\xi_t\}$ *with values in R satisfies the mixing condition* (5) *and, for some* $\delta > 0$, $E|\xi_t|^{2+\delta} < \infty$. *If for some* $\varepsilon > 0$ *the series*

$$\sum_{n=1}^{\infty} n^{1+\varepsilon} \alpha^{\delta/(2+\delta)}(n) < \infty$$

If $\sigma^2 \neq 0$, *then the process satisfies the CLT with* $f(x) = x$, $x \in R$.

4. We state now some necessary estimates for the covariance of random variables.

LEMMA 1. *Suppose that a stationary random process* $\{\xi_t\}$ *satisfies the g.m.c.,* $V_1 = (-k, 0)$, $V_2 = (n, n+m)$ *and* $\xi_I = (\xi_t, t \in I)$, $I \subset \mathbf{Z}$. *Let the functions* $\varphi_i(x_t, t \in V_i)$, $i = 1, 2$, *be continuous with respect to the metric* ϱ_{V_i}, $i = 1, 2$, *respectively, and* $\tau_{V_i}(\gamma)$, $i = 1, 2$, *be their continuity moduli. Suppose also that, for some* $s, u > 1$ $(1/s + 1/u < 1)$ *the moments* $\mathrm{E}|\varphi_1(\xi_{V_1})|^s$ *and* $\mathrm{E}|\varphi_2(\xi_{V_2})|^u$ *exist. Then, for any* $\gamma > 0$,

$$(4)\quad |\mathrm{E}\varphi_1(\xi_{V_1})\varphi_2(\xi_{V_2}) - \mathrm{E}\varphi_1(\xi_{V_1})\mathrm{E}\varphi_2(\xi_{V_2})|$$
$$\leqslant \tau_{V_1}(\gamma)\mathrm{E}|\varphi_2(\xi_{V_2})| + \tau_{V_2}(\gamma)\mathrm{E}|\varphi_1(\xi_{V_1})| +$$
$$+ B\mathrm{E}^{1/s}|\varphi_1(\xi_{V_1})^s \mathrm{E}^{1/u}|\varphi_2(\xi_{V_2})|^u (\gamma^{-1}\alpha_\varrho(n))^{1-1/s-1/u}, \quad 0 < B < \infty.$$

If, with probability 1, $|\varphi_i(\xi_{V_i})| \leqslant C_i < \infty$, $i = 1, 2$, *then the right-hand side of* (4) *may be replaced by*

$$(4')\quad \tau_{V_1}(\gamma)\mathrm{E}|\varphi_2(\xi_{V_2}) + \tau_{V_2}(\gamma)\mathrm{E}|\varphi_1(\xi_{V_1})| + \frac{2C_1C_2\alpha_\varrho(n)}{\gamma}.$$

Proof. Let $\varphi(x_t, t \in V_1 \cup V_2) = \varphi_1(x_t, t \in V_1)\varphi_2(x_t, t \in V_2)$. Suppose the random vector $\eta_{V_1 \cup V_2} = (\eta_t, t \in V_1 \cup V_2)$ has the distribution $P_{V_1} \times P_{V_2}$, $P_{V_i} \in \boldsymbol{P}$, $i = 1, 2$,

$$A_\gamma = \{\varrho_{V_1 \cup V_2}(\xi_{V_1 \cup V_2}, \eta_{V_1 \cup V_2}) < \gamma\},$$

$\bar{A}_\gamma$ is the complement of A_γ and, with probability 1,

$$|\varphi_i(\xi_t, t \in V_i)| \leqslant C_i < \infty, \quad i = 1, 2.$$

Then

$$|\mathrm{E}\varphi_1(\xi_{V_1})\varphi_2(\xi_{V_2}) - \mathrm{E}\varphi_1(\xi_{V_1})\mathrm{E}\varphi_2(\xi_{V_2})|$$
$$= |\mathrm{E}\varphi(\xi_{V_1 \cup V_2}) - \mathrm{E}\varphi(\eta_{V_1 \cup V_2})|$$
$$\leqslant \mathrm{E}_{A_\gamma}|\varphi(\xi_{V_1 \cup V_2}) - \varphi(\eta_{V_1 \cup V_2})| + \mathrm{E}_{\bar{A}_\gamma}|\varphi(\xi_{V_1 \cup V_2}) - \varphi(\eta_{V_1 \cup V_2})|$$
$$\leqslant \mathrm{E}_{A_\gamma}|\varphi_1(\xi_{V_1})||\varphi_2(\xi_{V_2}) - \varphi_2(\eta_{V_2})| + \mathrm{E}_{A_\gamma}|\varphi_2(\eta_{V_2})||\varphi_1(\eta_{V_1}) - \varphi_1(\xi_{V_1})| + \frac{2C_1C_2}{\gamma}\alpha_\varrho(n)$$
$$\leqslant \tau_{V_2}(\tau)\mathrm{E}|\varphi_1(\xi_{V_1})| + \tau_{V_1}(\gamma)\mathrm{E}|\varphi_2(\xi_{V_2})| + \frac{2C_1C_2}{\gamma}\alpha_\varrho(n).$$

Thus inequality (4′) is proved [1].

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[1] We acknowledge that the idea of this inequality should be attributed to Dobrushin (see inequality (3.8) in [3]; note that inequality (3.8) contains a misprint: γ and $\delta(\gamma)$ should be interchanged).

Let us prove now inequality (4). Let

$$\varphi_i^{K_i}(x) = \begin{cases} \varphi_i(x) & \text{if } |\varphi_i(x)| \leqslant K_i, \\ K_i & \text{if } \varphi_i(x) > K_i, \\ -K_i & \text{if } \varphi_i(x) < -K_i, \end{cases}$$

$$K_i \in R_+,\ \bar{\varphi}_i^{K_i}(x) = \varphi_i(x) - \varphi_i^{K_i}(x), \quad x \in X^{|V_i|},\ i = 1, 2,$$

$\tau_i^{K_i}(\gamma)$ be the continuity modulus of $\varphi_i^{K_i}(x)$. It is easy to see that

$$\tau_i^{K_i}(\gamma) \leqslant \tau_{V_i}(\gamma), \quad |\varphi_i^{K_i}(x)| \leqslant K_i, \quad i = 1, 2.$$

Further,

$$\begin{aligned} &|\mathrm{E}\varphi_1(\xi_{V_1})\varphi_2(\xi_{V_2}) - \mathrm{E}\varphi_1(\xi_{V_1})\mathrm{E}\varphi_2(\xi_{V_2})| \\ &\leqslant \mathrm{E}\,|\varphi_1^{K_1}(\xi_{V_1})\varphi_2^{K_2}(\xi_{V_2}) - \varphi_1^{K_1}(\eta_{V_1})\varphi_2^{K_2}(\eta_{V_2})| + \mathrm{E}\,|\varphi_1^{K_1}(\xi_{V_1})\bar{\varphi}_2^{K_2}(\xi_{V_2}) +\\ &\quad + \mathrm{E}\,|\bar{\varphi}_1^{K_1}(\xi_{V_1})\varphi_2^{K_2}(\xi_{V_2})| + \mathrm{E}\,|\bar{\varphi}_1^{K_1}(\xi_{V_1})\bar{\varphi}_2^{K_2}(\xi_{V_2})| + \\ &\quad + \mathrm{E}\,|\varphi_1^{K_1}(\xi_{V_1})|\,\mathrm{E}\,|\bar{\varphi}_2^{K_2}(\xi_{V_2})| + \mathrm{E}\,|\bar{\varphi}_1^{K_1}(\xi_{V_1})|\,\mathrm{E}\,|\varphi_2^{K_2}(\xi_{V_2})| + \\ &\quad + \mathrm{E}\,|\bar{\varphi}_1^{K_1}(\xi_{V_1})|\,\mathrm{E}\,|\bar{\varphi}_2^{K_2}(\xi_{V_2})|. \end{aligned}$$

Now it is enough to put

$$K_1 = \left(\frac{\gamma \mathrm{E}\,|\varphi_1(\xi_{V_1})|^s}{\alpha_\varrho(n)}\right)^{1/s}, \quad K_2 = \left(\frac{\gamma \mathrm{E}\,|\varphi_2(\xi_{V_2})|^u}{\alpha_\varrho(n)}\right)^{1/u}$$

and by proceeding in the same way as in [2] (§ 2, p. 390) one can prove (4).

In the sequel the following statement will be important:

LEMMA 2. *Let* $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ *be a vector such that*

$$|\mathrm{E}\prod_{s=i}^{n}\zeta_s| < \infty,\ i = 1, 2, \ldots, n-1; \quad |\mathrm{E}\zeta_i| \leqslant 1,\ i = 1, 2, \ldots, n.$$

Then

$$\begin{aligned} (5)\quad &|\mathrm{E}\prod_{s=1}^{n}\zeta_s - \prod_{s=1}^{n}\mathrm{E}\zeta_s| \\ &\leqslant \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|\mathrm{E}(\zeta_i-1)(\zeta_j-1)\prod_{s=j+1}^{n}\zeta_s - \mathrm{E}(\zeta_i-1)\mathrm{E}(\zeta_j-1)\prod_{s=j+1}^{n}\zeta_s|. \end{aligned}$$

Proof. It is well-known ([2], § 4, p. 429) that, under the conditions of Lemma 2,

$$(6)\qquad |\mathrm{E}\prod_{s=1}^{n}\zeta_s - \prod_{s=1}^{n}\mathrm{E}\zeta_s| \leqslant \sum_{i=1}^{n-1}|\mathrm{E}\zeta_i\prod_{s=i+1}^{n}\zeta_s - \mathrm{E}\zeta_i\mathrm{E}\prod_{s=i+1}^{n}\zeta_s|.$$

We can write

$$|\mathrm{E}\zeta_i \prod_{s=i+1}^{n} \zeta_s - \mathrm{E}\zeta_i \mathrm{E} \prod_{s=i+1}^{n} \zeta_s|$$

$$= |\mathrm{E}(\zeta_i - 1)\zeta_{i+1} \prod_{s=i+2}^{n} \zeta_s - \mathrm{E}(\zeta_i - 1)\mathrm{E}\zeta_{i+1} \prod_{s=i+2}^{n} \zeta_s|$$

$$\leqslant |\mathrm{E}(\zeta_i - 1)(\zeta_{i+1} - 1) \prod_{s=i+2}^{n} \zeta_s - \mathrm{E}(\zeta_i - 1)\mathrm{E}(\zeta_{i+1} - 1) \prod_{s=i+2}^{n} \zeta_s| +$$

$$+ |\mathrm{E}(\zeta_i - 1)\zeta_{i+2} \prod_{s=i+3}^{n} \zeta_s - \mathrm{E}(\zeta_i - 1)\mathrm{E}\zeta_{i+2} \prod_{s=i+3}^{n} \zeta_s.$$

Continuing this procedure we obtain

$$(7) \quad |\mathrm{E}\zeta_i \prod_{s=i+1}^{n} \zeta_s - \mathrm{E}\zeta_i \mathrm{E} \prod_{s=i+1}^{n} \zeta_s|$$

$$\leqslant \sum_{j=i+1}^{n} |\mathrm{E}(\zeta_i - 1)(\zeta_j - 1) \prod_{s=j+1}^{n} \zeta_s - \mathrm{E}(\zeta_j - 1)\mathrm{E}(\zeta_j - 1) \prod_{s=j+1}^{n} \zeta_s|.$$

Substituting (7) into (6) we get Lemma 2.

5. Now we are going to prove our theorem.

In the sequel $p = p(n)$ and $q = q(n)$, $n \in N$, denote the positive integer-valued functions.

LEMMA 3. *Let* $\{\eta_t\}$ *be a real-valued stationary random process such that* $\mathrm{E}\eta_t^2 < \infty$ *and:*

1. $\mathscr{D}S_n \sim cn$, $n \to \infty$, *where* $0 < c < \infty$, $S_n = \sum_{t=1}^{n} \eta_t$;

2. *for any function* $p = p(n)$, $p(n) \to \infty$, $p = o(n)$, $n \to \infty$, *there exists a function* $q = q(n)$, $q(n) \to \infty$, $q = o(p)$, $n \to \infty$, *such that, for every real* t,

$$|\mathrm{E} \prod_{j=1}^{k} \exp\{it\hat{S}_p^{(j)}\} - \prod_{j=1}^{k} \mathrm{E}\exp\{it\hat{S}_p^{(j)}\}| \to 0, \qquad n \to \infty,$$

where

$$\hat{S}_p^{(j)} = (\mathscr{D}S_n)^{-1/2} S_p^{(j)}, \quad S_p^{(j)} = \sum_{\substack{s=(j-1)p+\\+(j-1)q+1}}^{jp+(j-1)q} (\eta_s - \mathrm{E}\eta_s), \quad j = 1, 2, \dots, k$$

and $k = k(n) = [n/(p+q)]$.

Then for this process the CLT with identity function f jolds.

Proof. It is clear that there exists a function $p = p(n)$ such that $(\mathscr{D}S_p^{(1)})^{-1} \int (S_p^{(1)})^2 \, dP \to 0$ as $n \to \infty$, integrating for $|S_p^{(1)}| \geqslant \varepsilon \sqrt{\mathscr{D}S_n}$, where $\varepsilon > 0$, $p(n) \to \infty$, $p = o(n)$, $n \to \infty$.

Now, to complete the proof, it remains to apply the Bernstein method ([2], § 4, p. 426) for this $p = p(n)$.

Thus, in order to prove our theorem it is sufficient to verify the conditions of Lemma 3 for the process $\{\eta_t\} = \{f(\xi_t)\}$.

Let us verify condition 1. We have

$$\mathscr{D}\Big(\sum_{t=1}^{n} f(\xi_t)\Big) = \sum_{t,s=1}^{n} \big(\mathrm{E}f(\xi_t)f(\xi_s) - \mathrm{E}f(\xi_t)\mathrm{E}f(\xi_s)\big)$$

$$= n\mathrm{E}\big(f(\xi_1) - \mathrm{E}f(\xi_j)\big) + 2\sum_{t=2}^{n}(n-t+1)\big(\mathrm{E}f(\xi_1)f(\xi_t) - \mathrm{E}f(\xi_1)\mathrm{E}f(\xi_t)\big)$$

and

$$\lim_{n\to\infty} n^{-1}\mathscr{D}\Big(\sum_{t=1}^{n} f(\xi_t)\Big) = \mathrm{E}f(\xi_1) - \mathrm{E}f(\xi_1)^2 + 2\lim_{n\to\infty}\sum_{t=2}^{n}\big(\mathrm{E}f(\xi_1)f(\xi_t) -$$
$$- \mathrm{E}f(\xi_1)\mathrm{E}f(\xi_t)\big) - 2\lim_{n\to\infty} n^{-1}\sum_{t=2}^{n} t\big(\mathrm{E}f(\xi_1)f(\xi_t) - \mathrm{E}f(\xi_1)\mathrm{E}f(\xi_t)\big). \tag{8}$$

By Lemma 1 we get

$$|\mathrm{E}f(\xi_1)f(\xi_t) - \mathrm{E}f(\xi_1)\mathrm{E}f(\xi_t)| \leqslant 2C\tau(\gamma_t) + C\left(\frac{\alpha_\varrho(t)}{\gamma_t}\right)^{\delta/(2+\delta)}, \qquad 0 < C < \infty,$$

hence

$$\sigma_f^2 \leqslant \mathrm{E}f^2(\xi_1) + 2C\sum_{t=1}^{\infty}\tau(\gamma_t) + 2C\sum_{t=1}^{\infty}\beta^{\delta/(2+\delta)}(t).$$

The second summand in (8) vanishes as $n \to \infty$ by the well-known Kronecker lemma.

It remains to check condition 2. Let

$$W_t(x) = \exp\Big\{itB\sum_{s=1}^{m} f(x_s)\Big\} - 1, \qquad m \in N,\ 0 < B < \infty,\ x \in X^m,$$

X^m being a metric space with metric (2). Since

$$|W_t(x) - W_t(\tilde{x})| \leqslant B|t|\sum_{s=1}^{m}|f(x_s) - f(\tilde{x}_s)|, \qquad x, \tilde{x} \in X^m,$$

we conclude that the continuity modulus of the function $W_t(x)$ does not exceed $B|t|m\tau^f(\gamma)$, where $\tau^f(\gamma)$ is the continuity modulus of f. By Lemma 1 for $j > r$ and $s = u = 2+\delta$, $\delta > 0$, we have

(9) $$|\mathrm{E}(\exp\{it\hat{S}_p^{(r)}\}-1)(\exp\{it\hat{S}_p^{(j)}\}-1)\prod_{s=j+1}^{k}\exp\{it\hat{S}_p^{(s)}\}-$$

$$-\mathrm{E}(\exp\{it\hat{S}_p^{(r)}\}-1)\,\mathrm{E}(\exp\{it\hat{S}_p^{(j)}\}-1)\prod_{s=j+1}^{k}\exp\{it\hat{S}_p^{(s)}\}|$$

$$\leqslant B_1\frac{|t|}{\sqrt{n}}\tau^f(\gamma_{(j-\tau)q})\,\mathrm{E}\,|\exp\{it\hat{S}_p^{(1)}\}-1|+$$

$$+B_2\,\mathrm{E}^{2/(2+\delta)}|\exp\{it\hat{S}_p^{(1)}\}-1|^{2+\delta}\left(\frac{\alpha_\varrho((j-r)q)}{\gamma_{(j-\tau)q}}\right)^{\delta/(2+\delta)}$$

$$\leqslant B_3\left[\frac{|t|\,p\sqrt{p}}{n}\tau^f(\gamma_{(j-r)q})+\frac{p^2}{n}\left(\frac{\alpha_\varrho((j-r)q)}{\gamma_{(j-r)q}}\right)^{\delta/(2+\delta)}\right],\qquad 0<B_i<\infty,\ i=1,2,3.$$

By Lemma 2 and (9) we get

$$|\mathrm{E}\prod_{j=1}^{k}\exp\{it\hat{S}_p^{(j)}\}-\prod_{j=1}^{k}\mathrm{E}\exp\{it\hat{S}_p^{(j)}\}|$$

$$\leqslant B_4\Big(|t|\frac{p\sqrt{p}}{n}\frac{n}{p}\sum_{j=1}^{\infty}\tau^f(\gamma_{jq})+\frac{p^2}{n}\frac{n}{p}\sum_{j=1}^{\infty}\left(\frac{\alpha_\varrho(jq)}{\gamma_{jq}}\right)^{\delta/(2+\delta)}$$

and then

$$|\mathrm{E}\prod_{j=1}^{k}\exp\{it\hat{S}_p^{(j)}\}-\prod_{j=1}^{k}\mathrm{E}\exp\{it\hat{S}_p^{(j)}\}\leqslant B_4\Big(|t|\sqrt{p}\sum_{j=1}^{\infty}\tau^f(\gamma_{jq})+p\sum_{j=1}^{\infty}\beta^{\delta/(2+\delta)}(jq)\Big).$$

The monotonicity of the members of this series implies

$$\tau(\gamma_{jq})\leqslant\frac{2}{q}\sum_{k\geqslant(j-1/2)q}^{jq}\tau(\gamma_k),$$

$$\beta^{\delta/(2+\delta)}(jq)\leqslant\frac{2}{q}\sum_{k\geqslant(j-1/2)q}\beta^{\delta/(2+\delta)}(k),\qquad j=1,2,\ldots,$$

hence

$$\sum_{j=1}^{\infty}\tau(\gamma_{jq})\leqslant\frac{2}{q}\sum_{j\geqslant q/2}^{\infty}\tau(\gamma_j),\qquad \sum_{j=1}^{\infty}\beta^{\delta/(2+\delta)}(jq)\leqslant\frac{2}{q}\sum_{j\geqslant q/2}^{\infty}\beta^{\delta/(2+\delta)}(j).$$

Finally,

(10) $$|\mathrm{E}\prod_{j=1}^{k}\exp\{it\hat{S}_p^{(j)}\}-\prod_{j=1}^{k}\mathrm{E}\exp\{it\hat{S}_p^{(j)}\}|$$

$$\leqslant B_4|t|\frac{2\sqrt{p}}{q}\sum_{j\geqslant q/2}^{\infty}\tau(\gamma_j)+\frac{2p}{q}\sum_{j\geqslant q/2}^{\infty}\beta^{\delta/(2+\delta)}(j),$$

as it is obvious that one can choose the function $q(n) \to \infty$, $q = o(p)$, $n \to \infty$, such that the right-hand side of (10) tends to zero as $n \to \infty$.

Acknowledgement. The author expresses his gratitude to Professor R. V. Ambartzumian and Professor R. L. Dobrushin for their useful comments and suggestions.

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Received on 9. 9. 1986